Maximum Likelihood bounds for channel independent LP-OFDM systems over correlated channels

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Abstract — **Classes of channel independent Linear Precoded (LP-)OFDM systems are finding application in modern wireless communication protocols such as LTE. There is a need to provide accurate and simple Maximum Likelihood (ML) error rate bounds for the evaluation of these systems. In this paper we extend our previous work in this area by providing results applicable to the more realistic case of correlated fading. Two approaches are adapted, firstly the specific case of the uniform power delay profile is considered and secondly a less tight bound is developed for arbitrary power delay profiles.**

Keywords — **Equalization, Vector Systems, ML bounds, OFDM, Linear Precoding, Correlated Channels**

I Introduction

Orthogonal Frequency Domain Modulation (OFDM) has found application in many of today's modern communication systems due mainly to its high spectral efficiency. In the presence of a frequency selective channel an OFDM transmission can be tailored to optimize the capacity of the link provided the transmitter has accurate Channel State Information (CSI). In the absence of CSI at the transmitter the conventional approach is to use a combination of coding and interleaving to mitigate the effect of channel imperfections.

The authors of [1] proposed a system known as Linear Precoded (LP-)OFDM where the source symbols are spread across all sub-carriers using a linear matrix pre-multiplication step. Building on this work the authors of [2] showed that if the precoding matrix is selected to be a) unitary, and b) each entry having equal magnitude, then the Bit Error Rate (BER) over an unknown channel is, on average, minimized. This class of channel independent LP-OFDM system has found application in the 4^{th} generation mobile standard[1].

In this paper we consider the Bit and Symbol Error Rate (BER, SER) of such systems over various statistical channel models. In particular our previous work which found Maximum Likelihood lower bounds for an un-correlated fading channel model is extended to cover the more realistic (and important) case of correlated fading.

The structure of this paper is as follows:

In section II a review of selected results previously derived is presented as they will be used as a foundation for the examination of the correlated fading case considered in section III. Two approaches are then taken. In section IV bounds are derived for the case of a channel with a uniform power delay profile. In section V the more general case of channels with arbitrary power delay profiles is considered. Section VI provides some simulation results in support of the newly developed bounds for the uniform channel case.

[1] Long Term Evolution (LTE) uses Single Carrier SC-OFDM on the uplink which belongs to this class of channel independent LP-OFDM systems.

II Review of previous work

In [3] we considered the channel independent LP-OFDM system with K sub-carriers each modulated with symbols having variance σ_x^2 and corrupted with Additive White Gaussian Noise (AWGN) with variance σ_n^2 (corresponding to a flat two-sided noise Power Spectrum Density (PSD) with magnitude ${}^{N_o}/2$). The frequency domain channel gains are given by the set of K complex quantities $\left\{\tilde{h}_k\right\}$, and the resulting lower bound on the M-ary QAM SER, $\mathrm{P}_\mathrm{e}^{\mathrm{ML}}$, was found to be:

$$\mathrm{P}_\mathrm{e}^{\mathrm{ML}} \geqslant 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3\gamma_{\mathrm{in}}}{(M-1)}\frac{1}{K}\sum_{k=1}^{K}\left|\tilde{h}_k\right|^2}\right) \tag{1}$$

where $Q(x) \triangleq \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{\frac{-t^2}{2}}dt$.
and $\gamma_{in} \triangleq \frac{\sigma_x^2}{\sigma_n^2}$ is a measure of Signal to Noise Ratio (SNR).
The corresponding lower bound for the QPSK BER, P_b, is:

$$\mathrm{P}_\mathrm{b}^{\mathrm{ML}} \geqslant Q\left(\sqrt{2\frac{E_b}{N_0}\frac{1}{K}\sum_{k=1}^{K}\left|\tilde{h}_k\right|^2}\right) \tag{2}$$

where E_b is the energy per bit.

Both bounds depend on $\frac{1}{K}\sum_{k=1}^{K}\left|\tilde{h}_k\right|^2$ i.e. not on any one channel gain in particular, but rather an average of them all. In [3] we averaged these bounds over all channel instances under the assumption that each of the $\left\{\left|\tilde{h}_k\right|\right\}$ were independent and identically distributed (i.i.d) Rayleigh distributed Random Variables (RVs). This channel assumption, termed the *maximum diversity fading* channel model, is somewhat unrealistic as typically the power gain of neighboring sub-carriers in an OFDM system are highly correlated, i.e. they are identically distributed (Rayleigh), but they are not independent.

In this paper we now consider the more realistic case of a correlated faded channel, i.e. one where the sub-carriers are correlated.

III Correlated Rayleigh fading

Whilst the error rate equations (1) and (2) are expressed in terms of the frequency domain gains $\left\{\left|\tilde{h}_k\right|\right\}$, the channel is, in many wireless standards, specified as a time domain power delay profile. According to this description the channel is modeled as a FIR filter with K complex Gaussian distributed independent taps $\{h_k\}$, i.e. Rayleigh distributed magnitudes. Each tap represents a different path between transmitter and receiver and each is defined to have an associated average power gain $\{p_k\}$ which are collectively known as the *power delay profile.*

It is shown in [4] that the Fourier transform of this power delay profile results in the autocorrelation function of the frequency response. Therefore if all $\{p_k\}$ are equal the autocorrelation is a delta function resulting in the maximum diversity fading channel model as before. Conversely if not all the $\{p_k\}$ are equal then the frequency domain channel contains correlated terms.

a) Expressing error rates in terms of $\{h_k\}$

Noting that the $\left\{\left|\tilde{h}_k\right|\right\}$ are the unitary DFT of the time domain channel taps $\{|h_k|\}$ we have, by Parseval's theorem, the following relationship:

$$\Phi \triangleq \frac{1}{K}\sum_{k=1}^{K}\left|\tilde{h}_k\right|^2 = \frac{1}{K}\sum_{k=1}^{K}|h_k|^2$$

where Φ is a RV representing the average channel power gain with, as yet unknown, Probability Density Function (PDF) $f_\Phi(\phi)$ and, without loss of generality, can be assumed to have unity average power gain:

$$E[\Phi] = 1$$

By substitution of the definition of Φ into (1) the SER ML lower bound $\mathrm{P}_\mathrm{e}^{\mathrm{ML}}$ over a specific channel instance becomes:

$$\mathrm{P}_\mathrm{e}^{\mathrm{ML}} = 4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{(M-1)}\gamma_{\mathrm{in}}\Phi}\right) \tag{3}$$

Taking the average over all channel instances, or equivalently Φ, we can write the SER ML lower bound as:

$$\mathrm{P}_\mathrm{e}^{\mathrm{LB}} = \mathrm{E}\left[4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{(M-1)}\gamma_{\mathrm{in}}\Phi}\right)\right] = 4\left(1-\frac{1}{\sqrt{M}}\right)\int_{-\infty}^{+\infty} f_\Phi(\phi)\, Q\left(\sqrt{\frac{3}{(M-1)}\gamma_{\mathrm{in}}\phi}\right)d\phi$$

To progress further we need to develop expressions for the PDF $f_\Phi(\phi)$ for some power delay profiles. Two distinct approaches will be adapted firstly we will concentrate on a concrete channel power delay profile, namely the length L uniform power delay profile, and derive closed form bounds for this case. In the second approach we will make some additional simplifications for an arbitrary power delay profile and derive some looser, but nevertheless useful, bounds.

IV Length L uniform power delay profile

In this channel model the average magnitude of the first L channel taps are Rayleigh distributed and the remaining $K-L$ taps are zero. For this case we have:

$$\Phi = \frac{1}{K}\sum_{k=1}^{K}|h_k|^2 = \frac{1}{K}\sum_{k=1}^{L}|h_k|^2$$

which has a PDF proportional to a Chi-Squared RV with $2L$ degrees of freedom with the constant of proportionality being computed by incorporating the unity mean condition. The resulting PDF is:

$$f_\Phi(\phi) = \frac{L^L}{(L-1)!}\phi^{L-1}\exp(-L\phi) \quad \phi \geqslant 0$$

Using the Craig form [5] of the $Q(\cdot)$ function:

$$Q(\alpha) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\exp\left(-\frac{\alpha^2}{2\sin^2(\theta)}\right)d\theta \quad \alpha > 0$$

Our expression for the lower bound on $\mathrm{P}_\mathrm{e}^\mathrm{ML}$ becomes:

$$A\int_0^{+\infty}\int_0^{\frac{\pi}{2}}\phi^{L-1}\exp\left(-\left(L+\frac{\frac{3}{2(M-1)}\gamma_\mathrm{in}}{\sin^2(\theta)}\right)\phi\right)d\theta d\phi$$

where

$$A \triangleq \frac{4}{\pi}\frac{L^L}{(L-1)!}\left(1-\frac{1}{\sqrt{M}}\right)$$

is a temporary convenience factor.

Swapping the order of integration, and by making use of [6, eqn.(3.351-3)], we get

$$\mathrm{P}_\mathrm{e}^\mathrm{LB} = A\int_0^{\frac{\pi}{2}}(L-1)!\left(L+\frac{\frac{3}{2(M-1)}\gamma_\mathrm{in}}{\sin^2(\theta)}\right)^{-L}d\theta$$
$$4\left(1-\frac{1}{\sqrt{M}}\right)\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+\frac{1}{L}\frac{3}{2(M-1)}\gamma_\mathrm{in}}\right)^L d\theta \quad (4)$$

as per [7].

Following the same procedure for QPSK, an ML lower bound on the BER can be found as:

$$\mathrm{P}_\mathrm{b,QPSK}^\mathrm{LB} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+\frac{1}{L}\frac{E_b}{N_0}}\right)^L d\theta \quad (5)$$

The accuracy of this bound will be examined in section VI.

a) *Limiting case, as $K \to \infty$*

In [3] we showed that there exists a limiting form of an expression similar to the bounds (4) and (5) presented above, and so applying the same logic we have the following result:

$$\lim_{L\to\infty}\left[\mathrm{P}_\mathrm{e}^\mathrm{LB}\right] = 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{(M-1)}\gamma_\mathrm{in}}\right) \quad (6)$$

and

$$\lim_{L\to\infty}\left[\mathrm{P}_\mathrm{b,QPSK}^\mathrm{LB}\right] \geqslant Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$$

This represents the limiting performance where there are many paths from transmitter to receiver (with very small delays between each). Note, as was also pointed out in [3], that these formulations are exactly as per the standard SER and BER equations for transmission over a flat (Additive White Gaussian Noise) AWGN channel [8].

V Arbitrary power delay profile

The approach taken above is an example of where the procedure for computing ML bounds on the error rate performance can be computed for a particular power delay profile. The example provided was mathematically tractable due to the relatively simple structure of the power delay profile; in practice this profile is quoted based on physical channel measurements and there is very little structure present making the evaluation of closed form bounds very difficult. There is a requirement to have a methodology to compute bounds for arbitrary power delay profiles $\{p_k\}$.

Presented here is an approach based on an accurate approximation to the Q function that allows ML lower bounds to be computed for arbitrary channels.

a) *Chiani approximation of* $\mathrm{P}_\mathrm{e}^\mathrm{ML}$

The authors of [9] proposed the very accurate approximation to the erfc function:

$$\mathrm{erfc}(x) \approx \frac{1}{6}e^{-x^2}+\frac{1}{2}e^{-\frac{4}{3}x^2}$$

Relating this to the Q function we have:

$$Q(x) \approx \frac{1}{12}e^{-\frac{1}{2}x^2}+\frac{1}{4}e^{-\frac{2}{3}x^2}$$

and so the SER ML lower bound for a specific instance can be written as (from equation (3)):

$$\mathrm{P}_\mathrm{e}^\mathrm{ML} \approx \left(1-\frac{1}{\sqrt{M}}\right)\left[\frac{1}{3}e^{-\frac{1}{2}\beta\Phi}+e^{-\frac{2}{3}\beta\Phi}\right]$$

where $\beta \triangleq \frac{3}{(M-1)}\gamma_\mathrm{in}$.

We wish to average over all channel instances, or equivalently Φ, yielding:

$$\mathrm{P}_\mathrm{e}^\mathrm{LB} = \left(1-\frac{1}{\sqrt{M}}\right)\left[\frac{1}{3}\mathrm{E}\left[e^{-\frac{1}{2}\beta\Phi}\right]+\mathrm{E}\left[e^{-\frac{2}{3}\beta\Phi}\right]\right]$$

Looking, for the moment, at the expected value of $e^{-\alpha\Phi}$, where α is any positive constant we have:

$$\begin{aligned}\mathrm{E}\left[e^{-\alpha\Phi}\right] &= \mathrm{E}\left[e^{-\alpha\frac{1}{K}\sum_{k=1}^{K}|h_k|^2}\right] \\ &= \mathrm{E}\left[\prod_{k=1}^{K} e^{-\frac{\alpha}{K}|h_k|^2}\right]\end{aligned}$$

But, as each of the $|h_k|^2$ are independent, then so too are their exponentials and the expected value of the product is the product of the expected values:

$$\mathrm{E}\left[e^{-\alpha\Phi}\right] = \prod_{k=1}^{K}\mathrm{E}\left[e^{-\frac{\alpha}{K}|h_k|^2}\right]$$

Noting that the $|h_k|^2$ are proportional to a Chi-squared distributed RV (with two degrees of freedom), each with mean p_k , the PDF is

$$\begin{aligned}\frac{2}{p_k}|h_k|^2 &\sim \chi_2^2 \\ \Rightarrow f_{|h_k|^2}(x) &= \frac{1}{p_k}e^{-\frac{x}{p_k}}\end{aligned}$$

Using the relationship $\mathrm{E}[g(X)] = \int f_X(x)\, g(x)\, dx$, we have:

$$\begin{aligned}\mathrm{E}\left[e^{-\frac{\alpha}{K}|h_k|^2}\right] &= \int_0^{+\infty}\frac{1}{p_k}e^{-\frac{x}{p_k}}e^{-\frac{\alpha}{K}x}dx \\ &= \frac{1}{p_k}\int_0^{+\infty}e^{-\left(\frac{1}{p_k}+\frac{\alpha}{K}\right)x}dx \\ &= \frac{1}{p_k}\left(\frac{1}{p_k}+\frac{\alpha}{K}\right)^{-1} \\ &= \frac{1}{1+\alpha\frac{p_k}{K}}\end{aligned}$$

Thus $E\left[e^{-\alpha\Phi}\right]$ becomes:

$$\mathrm{E}\left[e^{-\alpha\Phi}\right] = \prod_{k=1}^{K}\frac{1}{1+\alpha\frac{p_k}{K}}$$

Substituting back into the expression for the SER lower bound, we get:

$$\begin{aligned}&\mathrm{P}_{\mathrm{e}}^{\mathrm{LB}} = \left(1-\frac{1}{\sqrt{M}}\right)\left(\frac{1}{3}\mathrm{E}\left[e^{-\frac{1}{2}\beta\Phi}\right]+\mathrm{E}\left[e^{-\frac{2}{3}\beta\Phi}\right]\right) \\ &= \left(1-\frac{1}{\sqrt{M}}\right)\left(\frac{1}{3}\prod_{k=1}^{K}\frac{1}{1+\frac{1}{2}\beta\frac{p_k}{K}}+\prod_{k=1}^{K}\frac{1}{1+\frac{2}{3}\beta\frac{p_k}{K}}\right)\end{aligned}$$

b) High SNR case

In the limit as the SNR, γ_{in}, becomes large (or equivalently as $\beta\to\infty$) we can make the assumption that, for non-zero p_k, the $\beta\frac{p_k}{K}\gg 1$ and so the products can be combined as follows:

$$\begin{aligned}&\lim_{\beta\to\infty}\left[\mathrm{P}_{\mathrm{e}}^{\mathrm{LB}}\right] = \left(1-\frac{1}{\sqrt{M}}\right)\frac{K}{\beta}\frac{13}{6}\prod_{k\in S_K}\frac{1}{p_k} \\ &= \left(1-\frac{1}{\sqrt{M}}\right)\frac{13}{9}K(M-1)\left(\frac{1}{\prod_{k\in S_K}p_k}\right)\frac{1}{\gamma_{in}}\end{aligned}$$

where S_K is that set of indexes k where the p_k are considered large[2].

Note that the limit is inversely related to the SNR, γ_{in}.

Note also that the limiting performance does not depend on the detail of the power delay profile $\{p_k\}$, but rather just on the inverse of the product $\prod p_k$ (over $k\in S_K$). On first sight this inverse relationship seems counter-intuitive. However by recalling that the p_k were scaled (without loss of generality) such that their Arithmetic mean A_P is unity, and by expressing the the product $\prod p_k$ as a geometric mean we can use the well known inequality for averages of non-negative numbers, $G_P\leqslant A_p$, as follow:

$$\begin{aligned}\prod p_k &= (G_P)^{|S_K|} \\ &\leqslant (A_P)^{|S_K|} = 1 \\ \Rightarrow\left(\prod p_k\right)^{-1} &\geqslant 1\end{aligned}$$

with equality occurring if and only if all the p_k participating in the product are equal (i.e. the uniform power delay profile). That is to say that average error rate is minimized for the uniform power delay and any deviation from uniform leads to an increases average error rate.

VI Simulation results

In order to verify the tightness of the newly derived lower bounds for the transmission of channel independent LP-OFDM over a correlated faded channel with a uniform power delay profile, a Sphere Decoder (which is an efficient ML implementation) was simulated and its performance along with the lower bounds are also shown in figure 1.

It can be seen from this figure, that the ML bounds are tightest for a larger number of independent fading paths (larger values of L), but even for modest channel lengths the bounds are none-the-less within a fraction of a dB.

VII Conclusion

In this paper several lower bounds on the Maximum Likelihood (ML) error rate performance for an important class of channel independent Linear Precoded (LP-)OFDM systems were developed.

This work builds on previous work for maximum diversity (non-correlated) Rayleigh fading channel models by addressing the more realistic correlated fading case. The approach was two fold; firstly the specific case of a channel with a uniform power delay profile was considered and bounds developed which were later verified by simulation.

[2] In this context, *large*, is understood that to sufficiently big so that $\beta\frac{p_k}{K}\gg 1$, and the p_k associated with those indexes not included in S_K are consider to be negligible, i.e. $p_k=0,\ \forall k\notin S_K$.

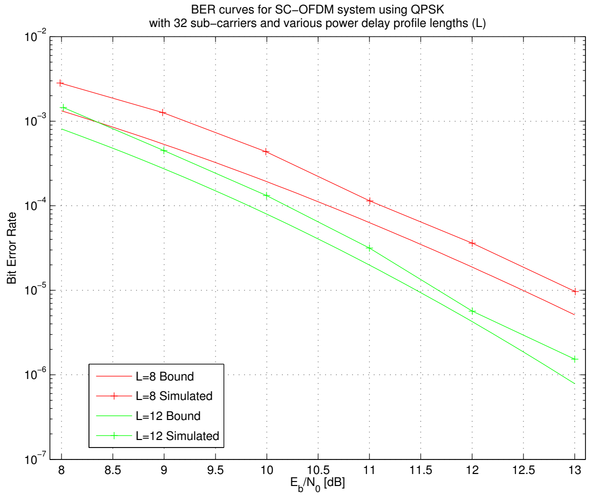

Figure 1: BER curves QPSK source symbols over a SC-OFDM system with 32 sub-carriers over a length L uniform power delay profile channel. Illustrated is both the ML lower bound, equation (5), and simulated BER performance for a Sphere Decoder for two values of L.

Secondly, a completely different approach was taken to develop bounds for channels with arbitrary power delay profiles leading to the discovery of a looser ML lower bound but with very wide applicability. It was noted that, under high SNR conditions, this bound does not depend on the specifics of the channel delay profile, but rather just on the geometric mean of power gains of the significant paths. Additionally it was shown that, of all the possible power delay profiles, the uniform profile performs best, with any deviation from uniform leading to an increased average error rate.

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